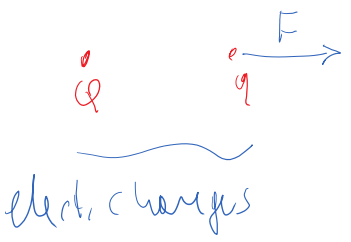


$$F = K \cdot q Q \, \frac{x(q) - x(Q)}{|x(q) - x(Q)|^3}$$

$$K = \frac{1}{4\pi\varepsilon_0}$$

$$E(x) = K Q \, \frac{x - x(Q)}{|x - x(Q)|^3}$$ electric field.

$q = 1$ in x

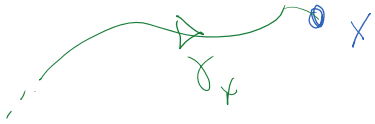
μ : distribution.

$$E^{\mu}(x) = K \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \, d\mu(y)$$

Obs. $\frac{x}{|x|^3} = -\nabla\left(\frac{1}{|x|}\right) \Rightarrow \vec{E}^{\mu} = -\nabla V^{\mu}$

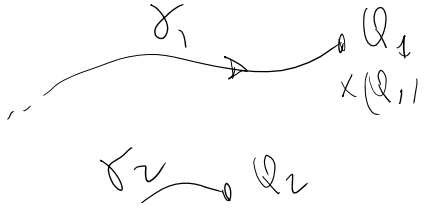
$$V^{\mu}(x) = K \int \frac{d\mu(y)}{|x - y|} = K \cdot \mu * \frac{1}{|x|} = V^{\delta_0} * \mu(x)$$

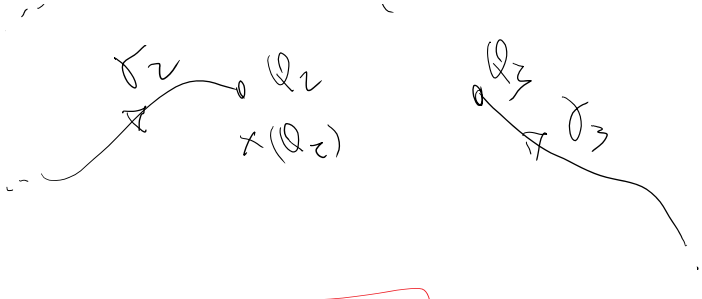
potential generated by μ



$$V^{\mu}(x) = -\int_{\gamma_x} E^{\mu}(y) \cdot dy$$

The energy $\frac{1}{2}\mathcal{E}$ of the distribution is the work needed to assemble the distribution.





$$\text{ENERGY} = \boxed{\frac{1}{2}\mathcal{E}(\mu)} = -Q_2 \int_{\gamma_2} E^{Q_1}(y)\cdot dy$$

$$= -Q_3\left[\int_{\gamma_2} E^{Q_2}(y)\cdot dy + \int_{\gamma_1} E^{Q_1}(y)\cdot dy\right] - \dots$$

$$= -\sum_{1=i<j\le N} Q_j \int_{\gamma_j} E^{Q_i}(y)\cdot dy = \boxed{\sum_{i<j} Q_j V^{Q_i}(x_j)}$$

$$= \boxed{\frac{K}{2}\sum_{i\neq j}\frac{Q_j Q_i}{|x_j - x_i|}} \qquad K = \frac{1}{4\pi\varepsilon_0}$$

$$\boxed{\mathcal{E}(\mu) = \frac{K}{2}\iint \frac{d\mu(x)\, d\mu(y)}{|x-y|}}$$

More information on V^μ. $(\cancel{\varepsilon_0 = 1})$.

THM. $\Delta V^\mu = -\frac{\mu}{\varepsilon_0}$. LAPLACE EQUATION

$V^\mu(\infty) = 0$ (if $\operatorname{supp}(\mu)$ is COMPACT).

$$\Delta = \partial_{x_1x_1} + \partial_{x_2x_2} + \partial_{x_3x_3},$$

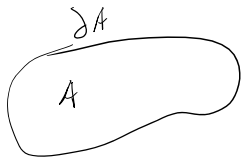
$\mathbb{R}^3 \ni x = r\omega$, $|\omega| = 1$ $\omega \in S^2$

$$\Delta = \partial_{rr} + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2}$$

$$\Downarrow$$

$$\Delta\left(\frac{1}{|x|}\right) = 0 \text{ for } x \neq 0.$$

$$\Delta\left(\frac{1}{|x|}\right) = 0 \quad \text{for } x \neq 0.$$



USE DIVERGENCE THEOREM:

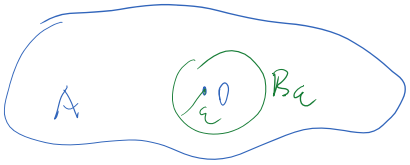
$$\int_{\partial A} v \cdot n \, d\sigma = \int_A \nabla \cdot v \, d\mathrm{vol}.$$

Lemma: $$\int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \cdot n \, d\sigma = \int_A \Delta\left(\frac{1}{|x|}\right) d\mathrm{vol} = -4\pi \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$$

"distributions" $$= -4\pi \, \delta_0(A) = -4\pi \int_A \delta_0 \, d\mathrm{vol}$$

$$\Delta\left(\frac{1}{|x|}\right) = -4\pi \cdot \delta_0$$

$0 \notin A$: $$\int_A \Delta\left(\frac{1}{|x|}\right) d\mathrm{vol} = \int_A 0 = 0$$



$0 \in A \Rightarrow$ $$0 = \int_{A \setminus B_\varepsilon} \Delta\left(\frac{1}{|x|}\right) d\mathrm{vol}$$

$$= \int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \cdot n \, d\sigma - \int_{\partial B_\varepsilon} \nabla\left(\frac{1}{|x|}\right) \cdot n \, d\sigma$$

$$= \int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \cdot n \, d\sigma - \underbrace{\int_{|x|=\varepsilon} \frac{\partial}{\partial r}\left(\frac{1}{r}\right) d\sigma}_{-\left(-\frac{4\pi\varepsilon^2}{\varepsilon^2}\right)}$$

$$= \int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \cdot n \, d\sigma + 4\pi$$

$$\Rightarrow -4\pi = \int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \cdot n \, d\sigma = \int_A \underbrace{\Delta\left(\frac{1}{|x|}\right)}_{\text{DISTRIBUTION}} d\mathrm{vol} \quad \forall A$$

$$\| $$

$$-4\pi \int_A \delta_0 \, d\mathrm{vol}$$

$$\Rightarrow \boxed{-4\pi\delta_0 = \Delta\left(\frac{1}{|x|}\right)}$$

$$\Rightarrow -4\pi\mu = -4\pi(\mu * \delta_0) = -4\pi\,\Delta\left(\frac{1}{|x|}\right) * \mu$$

$$= -4\pi\,\Delta\left(\frac{1}{|x|} * \mu\right)$$ - EXERCISE;

$$\boxed{\begin{cases} \Delta V^{\mu} = -\dfrac{\mu}{\varepsilon_0} \\ V^{\mu}(\infty) = 0 \end{cases}}$$

WE GOT RID OF GRADIENTS.

GETTING RID OF DERIVATIVES.

Recall: $\mathcal{E}(\mu) = k \iint \dfrac{d\mu(x)\, d\mu(y)}{|x-y|} =$ $V^{\mu}(x)$

$$= \int V^{\mu}(x)\, d\mu(x) \qquad -\frac{\mu}{\varepsilon_0} = \Delta V^{\mu}$$

$$(*) = -\varepsilon_0 \int V^{\mu}(y)\, \underbrace{\Delta V^{\mu}(y)\, dVol(y)}_{\text{MUSH.}}$$

DIV. THM.

$$= \boxed{\varepsilon_0 \int |\nabla V^{\mu}(x)|^2\, dVol\, x}$$

DIRICHLET INTEGRAL OF V^{μ}: CALCULUS OF VARIATIONS

$$= \varepsilon_0 \int V^{\mu}(y) \left[(-\Delta)^{1/2}(-\Delta)^{1/2} V^{\mu}(y)\right] dVol(y)$$

$(-\Delta)^{1/2} = ?$

$$= \varepsilon_0 \int (-\Delta)^{1/2} V^{\mu}(y) \cdot (-\Delta)^{1/2} V^{\mu}(y)\, dVol(y)$$

$$(*) = \varepsilon_0 \int \left|(-\Delta)^{1/2} V^{\mu}(y)\right|^2 dVol(y) :$$

$$\textcircled{B} = \varepsilon_0 \int |(-\Delta)^{1/2} V^{\mu}(y)|^2 \, dy$$

$$\boxed{\mathcal{E}(\mu) = \varepsilon_0 \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|} = \varepsilon_0 \cdot \int \left|(-\Delta)^{1/2} V^{\mu}\right|^2 dVol.}$$

EXERCISE: PROVE A = B USING FOURIER TRANSFORMS.

$$\hat{f}(\omega) = \int f(x)\, e^{-2\pi i x \cdot \omega}\, dx$$

$\omega \in \mathbb{R}^3$

$$f(x) = \int \hat{f}(\omega)\, e^{2\pi i x \cdot \omega}\, d\omega$$

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

$$\Rightarrow (-\Delta f)^{\wedge}(\omega) = 4\pi^2 |\omega|^2 \hat{f}(\omega)$$

DEF. $(-\Delta)^{\alpha} f = g$: $\hat{g}(\omega) = (4\pi^2 |\omega|^2)^{\alpha} \hat{f}(\omega)$

$\alpha > 0$

$$\boxed{(-\Delta)^{-1/2} f(x) \underset{\text{EXERCISE WITH F.T.}}{=} I_1 f(x) \overset{\mathbb{R}^3}{=} \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{f(y)\, dy}{|x-y|^2}}$$

$d\mu(y)$

RIESZ INTEGRAL.

- $\left[(-\Delta)^{-1/2} f\right]^{\wedge}(\omega) = \left[4\pi^2 |\omega|^2\right]^{-1/2} \hat{f}(\omega)$

$$= \frac{1}{2\pi |\omega|} \hat{f}(\omega)$$

- $(-\Delta)\left[(-\Delta)^{-1/2} (-\Delta)^{-1/2} f\right] = f$: LAPLACE EQUATION

$$(-\Delta)\left(I_1 I_1 f\right) = f \qquad (-\Delta) V^{\mu} = \frac{\mu}{\varepsilon_0}$$

i.e. $V^{\mu} = \frac{1}{\varepsilon_0} I_1 I_1 \mu$

$$\mathcal{E}(\mu) = \varepsilon_0 \cdot \int |(-\Delta)^{1/2} \mu|^2 \, dvol = \varepsilon_0 \cdot \int |I_1 \mu|^2 \, dvol$$

$$I_1 \mu(x) = \frac{1}{2\pi^2} \int \frac{d\mu(y)}{|x-y|^2}$$

V^{μ} and $\mathcal{E}(\mu)$ can be written as integrals: using the kernel $\frac{1}{2\pi^2} \frac{1}{|x-y|^2}$ I_1